Upward 'falling' jets and surface tension

By JOSEPH B. KELLER and MORTIMER L. WEITZ
Institute of Mathematical Sciences, New York University

(*Received* 1 *November* 1956)

According to the simple hydraulic theory of jets, each particle of a jet moves independently along a parabolic trajectory. Therefore a steady jet has a parabolic shape. We wish to consider how these results are modified by surface tension. For simplicity we will consider a two-dimensional jet of incompressible fluid.

The hydraulic theory is based upon the two assumptions, that the velocity is constant on each cross-section, and that the pressure is constant throughout the jet. Then the jet can be completely described in terms of its centre line $y(x, t)$, its vertical width $h(x, t)$, its horizontal velocity $u(x, t)$ and its vertical velocity $v(x, t)$. These four functions satisfy the following four differential equations, the first of which is a conservation of mass equation, the second a kinematic relation, and the last two are momentum equations.

$$\rho h_t + \rho(uh)_x = 0, \tag{1}$$

$$y_t + uy_x = v, \tag{2}$$

$$\rho h(u_t + uu_x) = -2Ty_x y_{xx}(1+y_x^2)^{-3/2}, \tag{3}$$

$$\rho h(v_t + uv_x) = 2Ty_{xx}(1+y_x^2)^{-3/2} - \rho gh. \tag{4}$$

In these equations ρ denotes the density of the liquid, T denotes the tension of the surface, and g denotes the acceleration of gravity.

Let us seek a steady (i.e. time-independent) solution of these equations. From (1) we obtain $uh = m$, where m is the constant flux of mass through any cross-section of the jet. Equation (2) yields $v = uy_x$, while (3) and (4) become

$$u_x = \frac{-2T}{m} y_x y_{xx}(1+y_x)^{-3/2}, \tag{5}$$

$$(uy_x)_x = \frac{2T}{m} y_{xx}(1+y_x^2)^{-3/2} - gu^{-1}. \tag{6}$$

Upon integrating (5) and denoting $u(0)$ by u_0, we obtain

$$u(x) = u_0 + \frac{2T}{m}[(1+y_x^2)^{-1/2} - (1+y_x^2(0))^{-1/2}]. \tag{7}$$

Now on using (7) to eliminate $u(x)$ from (6), we have

$$[1+\alpha(1+y_x^2)^{-1/2}]y_{xx} = -\beta. \tag{8}$$

Here the constants α and β are defined by

$$\alpha = \frac{2T}{mu_0 - 2T(1+y_x^2(0))^{-1/2}}, \tag{9}$$

$$\beta = \frac{gm^2}{[mu_0 - 2T(1+y_x^2(0))^{-1/2}]^2}. \tag{10}$$

Integration of (8) yields

$$y_x + \alpha \sinh^{-1} y_x = -\beta x + y_x(0) + \alpha \sinh^{-1} y_x(0). \tag{11}$$

The last equation is a first order equation for $y(x)$ involving three parameters. To reduce the number of parameters we set $y_x(0) = 0$, since the resulting curve will apply to any initial slope if read from that slope on. Next, we introduce the new variables $\beta x = \xi$ and $\beta y = \eta$. In terms of these variables (11) becomes

$$\eta_\xi + \alpha \sinh^{-1} \eta_\xi = -\xi. \tag{12}$$

This equation involves only the one parameter $\alpha = 2T/(mu_0 - 2T)$ and is therefore convenient for numerical integration.

When $\alpha = 0$, (12) yields the expected parabola $\eta = -\frac{1}{2}\xi^2$. In the figure graphs of the solution of the above equation are shown for $\alpha = 0$, 1 and -2. When the ξ, η variables are used, the curves with $\alpha > 0$ lie above the parabola obtained for $\alpha = 0$. On the other hand, when the x, y variables are used these curves lie below the parabola. Thus, as one expects, in the physical plane the jets with surface tension lie below the parabola, provided $\alpha > 0$, or $mu_0/2T > 1$.

However, when $\alpha < 0$, or $mu_0/2T < 1$, the jet rises instead of falling, even though it is initially projected horizontally (see the figure). It continues rising until all the initial kinetic energy is converted into potential and surface energy. Then the curvature and thickness become infinite and the theory fails. Presumably it would spill down before this point is reached. This phenomenon of a rising jet can occur only in a slow thin jet, since the above inequality requires that the kinetic energy $\frac{1}{2}\rho h u_0^2$ must be less than the surface energy T. In very crude experiments we have been unable to observe this behaviour. We believe this is because the rising flow is unstable at the necessarily low speed. Instead, a 'teapot-like' flow, in which the liquid runs along the lower surface of the spout, seems to be stable.

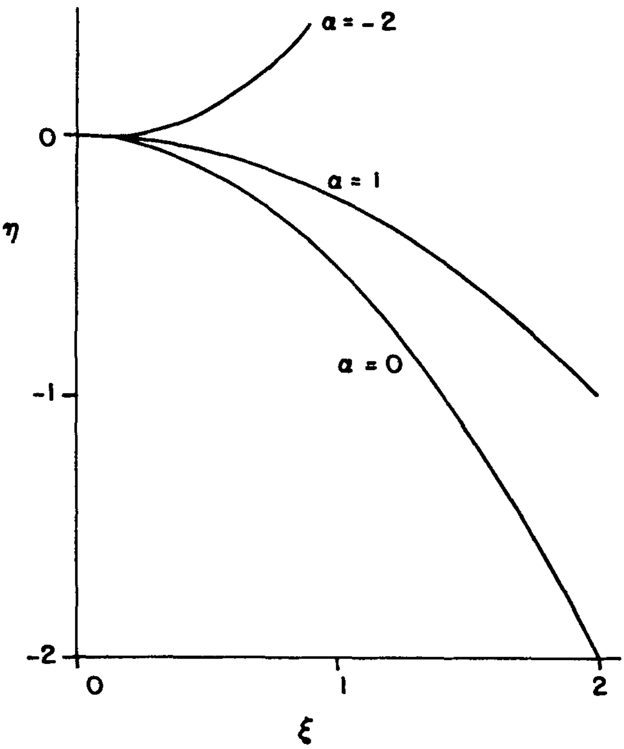
In order to understand the surprising phenomenon of a jet 'falling' upward, let us consider the simple problem of a falling body of mass M. If $y(t)$ denotes the height of the body measured positive upwards, then the equation of motion is $My_{tt} = -Mg$. From this equation we see that y_{tt} is negative. If we want the body to fall faster, we may push it down with a force $k^2 y_{tt}$ proportional to the acceleration. Then the equation of motion becomes

$$My_{tt} = -Mg + k^2 y_{tt},$$

and we now have

$$y_{tt} = -Mg/(M-k^2).$$

Thus as k^2 increases from zero, the acceleration becomes more negative, until $k^2 = M$, when there is no solution. For $k^2 > M$, y_{tt} is positive so the body falls upward. We may describe this example by saying that negative inertial mass of amount $-k^2$ has been added to the body. As more negative inertial mass is added, the body's downward acceleration continually increases until the total inertial mass $M-k^2$ becomes negative. Then the body falls upward.



Steady jets falling or rising under the influences of gravity and surface tension. The parabola $\alpha = 0$ occurs when surface tension is absent. For moderate surface tension ($\alpha > 0$) a curve such as that shown for $\alpha = 1$ occurs. In terms of the x, y variables this curve lies *below* the parabola. However, for larger surface tension ($\alpha < 0$) a curve like that shown for $\alpha = -2$ results. This curve represents an upward falling jet. The curves for $\alpha = 1$ and $\alpha = -2$ were obtained by numerical integration of equation (12).

This example is pertinent to the jet problem. To see this, note that near the nozzle the x-coordinate of a particle is approximately $u_0 t$. The surface tension force is approximately $2Ty_{xx} = 2Ty_{tt}/u_0^2$. Thus the above discussion applies to a particle of the jet with $k^2 = 2T/u_0^2$. We see that surface tension has the effect of adding a negative inertial mass to each particle without changing its gravitational mass.

These results appeared in "Thin unsteady heavy jets", by J. B. Keller and M. L. Weitz, Report IMM-NYU 186, Institute of Mathematical Sciences, New York University, 19 December 1952. They were also presented at the Ninth International Congress for Applied Mechanics, Brussels, September 1956.